Int. J. Solids Structures, 1974, Vol. 10, pp. 411–417. Pergamon Press. Printed in Gt. Britain.

ON THE DETERMINATION OF ELASTODYNAMIC CRACK TIP STRESS FIELDS

L. B. FREUND and J. R. RICE

Division of Engineering, Brown University, Providence, Rhode Island 02912, U.S.A.

(*Received* 29 *May* 1973)

Abstract—A linear elastic body in plane strain which contains a stationary crack and which is initially at rest and stress free is considered. It is shown that if the elastodynamic displacement field and stress intensity factor are known, as functions of crack length, for *any* symmetrical distribution of time-varying forces which acts on the body, subsequent to $t = 0$, then the stress intensity factor due to *any other* symmetrical load system whatsoever which acts on the same body may be directly determined. The other load system may be of arbitrary spatial distribution and time variation. Further, that part of the elastodynamic displacement field due to the other load system, which arises from the presence of the crack, may also be directly determined. The results are obtained by extension of Rice's mode of derivation of the corresponding Bueckner–Rice elastostatic results to Laplace-transformed elastodynamic variables. Likewise, the existence of a universal elastodynamic "weight function" for any given cracked body is demonstrated. As an application, Freund's recent result for the stress intensity factor due to suddenly applied concentrated forces on the crack surfaces is derived directly by our method, from de Hoop's earlier solution for suddenly applied uniform pressures.

INTRODUCTION

In a recent paper[1] Rice employed Irwin's relation between the crack tip energy release rate and the stress intensity factor, and also certain properties of perfect differentials, to show that if the displacement field and stress intensity factor are known as functions of crack length for any one symmetrical static load system acting on a linear elastic body in plane strain, then the stress intensity factor for any other symmetrical load system acting on the same body can be directly determined. Indeed, the entire displacement field due to introduction of the crack may likewise be determined for the other load system. The result leads concisely to the theory of a universal function for any given cracked body, known as the "weight function," the existence of which was previously demonstrated for isotropic bodies by Bueckner[2] on the basis of detailed considerations of analytic function theory.

Our purpose here is to extend Rice's method so as to derive similar results for dynamic stress fields in an elastic solid containing a crack. Consider a linear elastic solid containing a planar crack under conditions of plane strain. The body is assumed to be symmetrical with respect to the plane of the crack, and only loading systems resulting in the plane strain opening mode of deformation will be considered.

Consider any particular time-dependent loading on the body which contains a crack whose length is determined by the parameter l. For time $t < 0$, the material is stress free and at rest. At time $t = 0$, a traction distribution on the boundary Γ of the body and a body force distribution in the region A occupied by the body begin to act. The boundary traction and body force are assumed to be given at any later time $t > 0$ by $\mathbf{T} = Q_1 \mathbf{T}^{(1)}(\mathbf{x}, t)$ for $\mathbf{x}$ on Γ and by $\mathbf{F} = Q_1 \mathbf{F}^{(1)}(\mathbf{x}, t)$ for $\mathbf{x}$ in A, respectively. The parameter Q_1, which appears as a scale factor, will subsequently be viewed as a generalized force. Suppose that the resulting displacement field $\mathbf{U} = Q_1\mathbf{U}^{(1)}(\mathbf{x}, t)$ and stress intensity factor $K = Q_1 K^{(1)}(t)$ are known as functions of l. The main result here is in showing that this information is sufficient to determine the stress intensity factor for *any other load system whatsoever acting on the same body.* More precisely, it is shown that if the Laplace transforms on time of $\mathbf{U}$ and K are known as functions of l for all values of the Laplace transform parameter, then the Laplace transform on time of the stress intensity factor for any other load system can be determined. The convention is adopted whereby lower case symbols will represent Laplace transforms of functions defined by corresponding upper case letters. For example:

$$k(s) = \int_0^\infty \mathrm{e}^{-st}K(t)\,\mathrm{d}t. \tag{1}$$

ANALYSIS

A key step in the development of [1] was recognition of the fact that if W is the elastic strain energy per unit thickness of a statically loaded cracked body, then

$$(\partial W/\partial l)_{\text{fixed displ.}} = -H^{-1}K^2{}_{\text{static}} \tag{2}$$

where "fixed displacement" means that the derivative is taken with loaded portions of the boundary constrained against working displacements. For plane strain, H is given in terms of Young's modulus E and Poisson's ratio ν as $E/(1-\nu^2)$. Fortunately, a result analogous to (2) which applies in the case of dynamic loading may be determined. Let $\mathbf{u}$ be the Laplace transform of the displacement field arising from application of any surface traction and body force, with Laplace transforms $\mathbf{t}$ and $\mathbf{f}$, respectively. A functional Φ over the range of displacement fields, which is the analogue of W in the present case, is defined by

$$\Phi[\mathbf{u}; A'] = \int_{A'} \tfrac{1}{2}(\sigma_{ij}u_{i,j} + \rho s^2 u_i u_i - 2f_i u_i)\,\mathrm{d}A \tag{3}$$

where A' is any subregion of A, σ_{ij} is the stress matrix derived from u_i, and ρ is the mass density. Rice [3] has shown that the static elastic energy release rate can be expressed by the path independent J integral, which is a conservation law of the first type as discussed by Knowles and Sternberg [4], following from the potential energy variational theorem. A similar variational theorem based on Φ gives the elastodynamic field equations, and given the method for associating conservation laws with variational theorems outlined by Eshelby [5], a similar integral can be associated with Φ. This has been given by Nilsson [6], and is

$$J' = \int_{C'} [(\tfrac{1}{2}\sigma_{ij}u_{i,j} + \tfrac{1}{2}\rho s^2 u_i u_i - f_i u_i)\,\mathrm{d}y - \sigma_{ij}n_j u_{i,1}\,\mathrm{d}s] \tag{4}$$

where C' is the boundary of A' and $\mathbf{n}$ is the outward normal to C'. If A' is simply connected and if $\mathbf{f}$ is spatially uniform then $J' = 0$.

The result which is of primary interest here is that the J'-integral in (4) has the same value when taken along any path which begins on one face of the crack, surrounds the crack tip in question, and terminates on the opposite face. Following the analysis of [3], two main results can be deduced. First, by shrinking the path of integration onto the crack tip, it is seen that only those terms which are singular at the tip contribute to the value of J'. Because the singular terms are certain universal functions of position times the stress intensity factor k, it can be shown that $J' = k^2/H$. Second, by assuming that the region A' in (3) contains the crack tip and by differentiating with respect to l (recall that $\mathbf{u}$ is taken to be a known function of crack length), it can be shown that $J' = -\partial\Phi/\partial l$, where the boundary of A' is constrained against working displacements. In particular, if $A' = A$,

$$(\partial\Phi/\partial l)_{\text{fixed displ.}} = -H^{-1}k^2, \tag{5}$$

which is the desired result. Clearly, J' does not have the interpretation of energy release rate in this case, but rather the release rate of the energy-like quantity Φ defined on transformed variables. The same results apply when $\mathbf{f}$ is nonuniform, provided that J' is always interpreted in the limit as the path shrinks onto the crack tip.

From this point on, the analysis of the dynamic case parallels that of the static case, with Φ taking over the role of W. Consider two loading systems acting on the body which are characterized by the generalized forces Q_1 and Q_2. The complete solution to problem 1 is assumed to be known. Generalized displacements q_1 and q_2 are associated with any displacement field $\mathbf{u}$ by

$$q_i = \int_\Gamma \mathbf{t}^{(i)} \cdot \mathbf{u}\, \mathrm{d}\Gamma + \int_A \mathbf{f}^{(i)} \cdot \mathbf{u}\, \mathrm{d}A. \tag{6}$$

If both load systems are simultaneously applied, linear superposition yields

$$q_i = C_{ij} Q_j,\ C_{ij} = \int_\Gamma \mathbf{t}^{(i)} \cdot \mathbf{u}^{(j)}\, \mathrm{d}\Gamma + \int_A \mathbf{f}^{(i)} \cdot \mathbf{u}^{(j)}\, \mathrm{d}A, \tag{7}$$

where $Q_1\mathbf{u}^{(1)}$ and $Q_2\mathbf{u}^{(2)}$ are the separate transformed displacement fields. Finally, the Laplace transform of the stress intensity factor is defined in terms of the transformed stress by

$$k(s) = \lim_{x\to 0^+} (2\pi x)^{1/2}\sigma_{yy}(x, 0, s). \tag{8}$$

When both load systems are simultaneously applied to the same body, Φ may be regarded as a function of q_1, q_2 and l, with

$$\partial\Phi/\partial l = -k^2/H,\ \partial\Phi/\partial q_i = Q_i \tag{9}$$

where $k = Q_1 k^{(1)} + Q_2 k^{(2)}$. It is then possible to write the differential of Φ as

$$\delta\Phi = Q_1\, \delta q_i + Q_2\, \delta q_2 - (k^2/H)\, \delta l. \tag{10}$$

A transformation of variables yields the equivalent perfect differential

$$\delta(Q_1 q_1 + Q_2 q_2 - \Phi) = q_1\, \delta Q_1 + q_2\, \delta Q_2 + (k^2/H)\, \delta l. \tag{11}$$

Because (11) is a perfect differential, the following relations hold:

$$\partial q_i/\partial l \equiv (\mathrm{d}C_{ij}/\mathrm{d}l) Q_j = \partial(k^2/H)/\partial Q_i \equiv 2k^{(i)}k^{(j)}Q_j/H. \tag{12}$$

Since (12) must hold for arbitrary Q_1 and Q_2,

$$\mathrm{d}C_{ij}/\mathrm{d}l = 2k^{(i)}k^{(j)}/H. \tag{13}$$

But $C_{21} = C_{12}$ and $k^{(1)}$ are known functions of l. Thus, making use of (7), (13) can be solved for $k^{(2)}$ to yield

$$k^{(2)} = \frac{H}{2k^{(1)}} \left\{ \int_\Gamma \mathbf{t}^{(2)} \cdot \frac{\partial \mathbf{u}^{(1)}}{\partial l}\, \mathrm{d}\Gamma + \int_A \mathbf{f}^{(2)} \cdot \frac{\partial \mathbf{u}^{(1)}}{\partial l}\, \mathrm{d}A \right\}, \tag{14}$$

where $\partial u^{(1)}/\partial l$ is taken with $\mathbf{x}$ and s held fixed. This is our main result. It can be seen that the form of (14) is essentially the same as the corresponding result in equation (12) of [1].

AN APPLICATION

To illustrate the procedure of applying relation (14), consider the case in which A is the entire x, y-plane, with a crack along $y = 0$, $x < l$. In this case, Γ consists solely of the crack faces. The loading system 2 is defined by $\mathbf{F}^{(2)} = 0$ and $\mathbf{T}^{(2)}(x, \pm 0, s) = \pm\delta(x)H(t)\mathbf{e}_y$ where δ is the Dirac delta function and H is the unit step function. Thus, this loading system consists of suddenly applied concentrated normal forces acting on the crack faces at $x = 0$ and tending to open the crack, as shown in Fig. 1. This is one of the most recently solved dynamic elastic crack problems, and the results are reported in [7].

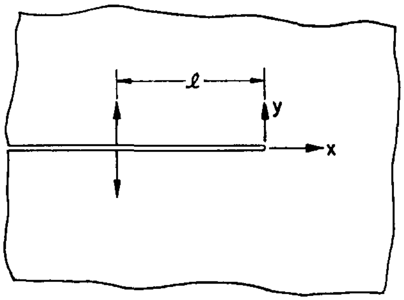

Fig. 1. The plane of deformation.

The Laplace transform of the stress intensity factor for loading system 2, $k^{(2)}$, will be determined according to relation (14) from the solution of the problem defined by the loading system $\mathbf{F}^{(1)} = 0$ and $\mathbf{T}^{(1)}(x, \pm 0, s) = \pm H(t)\mathbf{e}_y$ on Γ. The loading system 1 consists of a suddenly applied uniform normal pressure acting on the crack faces. This problem was first solved by de Hoop [8], who considered the general plane strain problem of diffraction of a longitudinal pulse by a half-plane crack. The loading system 1 corresponds to normal incidence of a step tension pulse.

The function $\partial u_y^{(1)}/\partial l$ is given along the crack faces $x < l$ by

$$\frac{\partial u_y^{(1)}}{\partial l}(x, \pm 0, s) = \frac{\mp D}{2\pi i s}\int_{B_2} \frac{(a - \xi)^{1/2}}{(\xi - c)S_-(\xi)}\, e^{s\xi(x-l)}\, d\xi \tag{15}$$

where a, b and c are the inverse longitudinal, shear and Rayleigh wave speeds. The path of integration B_2 is an infinite straight line parallel to the imaginary axis and lying in the strip $-a < Re(\xi) < a$ in the complex ξ-plane. $S_-(\xi)$ is the function, analytic in the left half of the ξ-plane, arising from the product factorization into sectionally analytic functions of the Rayleigh wave function in determining (15) by integral transform methods and the Wiener–Hopf technique. This function is analytic in the entire ξ-plane cut along the real axis from $\xi = a$ to $\xi = b$, and it is given by

$$S_-(\xi) = \exp\left\{-\frac{1}{\pi}\int_a^b \tan^{-1}\left[\frac{4\eta^2(\eta^2 - a^2)^{1/2}(b^2 - \eta^2)^{1/2}}{(2\eta^2 - b^2)^2}\right]\frac{d\eta}{\eta - \xi}\right\} \tag{16}$$

The function $(a - \xi)^{1/2}$ in the integrand of (15) is made single-valued by providing a branch cut along the real axis in the interval $a \le \mathrm{Re}\,\xi < \infty$, and the branch of positive square roots is chosen. Finally, the value of the constant D appearing in (15) is inconsequential for this development. The Laplace transform of the stress intensity factor for loading system 1 is given by

$$k^{(1)} = HD/(2^{1/2}s^{3/2}), \tag{17}$$

which yields the well-known result that $K^{(1)}$ itself increases from zero in proportion to $t^{1/2}$. By substituting (15) and (17) into (14), it is found that

$$k^{(2)} = \left(\frac{2}{s}\right)^{1/2} \frac{1}{2\pi i}\int_{B_2} \frac{(a-\xi)^{1/2}}{(\xi - c)S_-(\xi)}\, e^{-s\xi l}\, \mathrm{d}\xi. \tag{18}$$

In order to find the stress intensity factor itself, the Laplace transform (18) must be inverted. To this end, the path of integration B_2 is deformed into the right half of the ξ-plane so as to embrace the branch cut along the real axis. The integrand of (18) meets the conditions of Jordan's lemma, and no singularities of the integrand are crossed in so deforming B_2. Call the new path of integration B_2^*. Next, the inverse transform of (18) is written, the order of integration is changed, and the integral over s is evaluated to yield

$$\begin{aligned} K^{(2)}(t) &= \left(\frac{2}{\pi}\right)^{1/2} \frac{1}{2\pi i}\int_{B_2^*} \frac{H(t-\xi l)}{(t-\xi l)^{1/2}} \frac{(a-\xi)^{1/2}}{(\xi - c)S_-(\xi)}\, \mathrm{d}\xi \\ &= \frac{1}{2^{1/2}\pi^{3/2}} \int_a^{t/l} \mathrm{Im}\left[\frac{(a-\xi)^{1/2}}{(t-\xi l)^{1/2}(\xi - c)S_-(\xi)}\right] \mathrm{d}\xi. \end{aligned} \tag{19}$$

Both integral representations of the result (19) are considered in some detail in [7]. In particular, it is shown that for $t/l < b$ the second integral is most convenient, and it is evaluated numerically in [7]. On the other hand, for $t/l > b$ the first integral is more convenient, and it is evaluated by contour integration methods in [7]. The final result for $K^{(2)}(t)$ is shown in Fig. 2. The stress intensity factor is zero until the longitudinal wave arrives at $t = al$. It

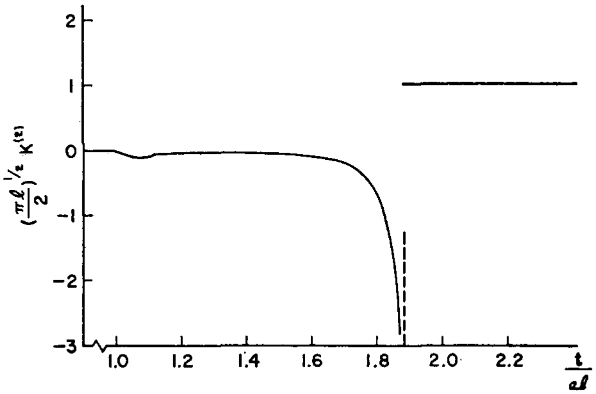

Fig. 2. Nondimensionalized stress intensity factor vs. t/al for suddenly applied loads on the crack faces.

remains negative as time increases, and it becomes square-root singular as the Rayleigh wave approaches. At the instant the Rayleigh wave arrives, $K^{(2)}$ takes on its appropriate static value $(2/\pi l)^{1/2}$. This value is maintained thereafter. The results are discussed in much greater detail in [7].

DISCUSSION

As in the static case, the stress intensity factor for loading system 2 can in no way depend on the particular choice of loading system 1. Hence, the function **h** analogous to the static weight function, defined in terms of any solution by

$$\mathbf{h} = \frac{H}{2k}\frac{\partial \mathbf{u}}{\partial l}(x, y, s), \tag{20}$$

is a universal function for a cracked body of any given shape, regardless of the way in which the body is loaded. Here $\partial\mathbf{u}/\partial l$ is taken with the loading history, as well as x, y and s, held fixed. The uniqueness of **h** can be established by following the analysis of [1]. Further, from (14) it is evident that if **t** and **f** are the transforms of any particular symmetrical loading system, then the corresponding transformed intensity factor is

$$k = \int_\Gamma \mathbf{t}\cdot\mathbf{h}\,\mathrm{d}\Gamma + \int_A \mathbf{f}\cdot\mathbf{h}\,\mathrm{d}A. \tag{21}$$

Also, once k is determined for this load system, one may solve for $\partial\mathbf{u}/\partial l$ from (20). In this way a knowledge of the weight function for each value of l enables that part of the displacement field arising from the presence of the crack to be determined by direct integration on l.

Our mode of presentation emphasizes the construction of the weight function from any one known solution. But it is also possible to directly set a boundary value problem for determination of **h**, as discussed by Bueckner[2] and Rice[1] in the static case. Indeed, it is clear from (20) that **h** satisfies the same field equations as does a dynamic displacement field, but that the "stresses" derived from **h** equilibrate zero boundary tractions and body forces. A non-zero field satisfies these homogeneous equations because **h** is not a member of the bounded-energy class of displacement fields for which elastic uniqueness results. Instead, **h** has a singular term of order $r^{-1/2}$, where r is distance from the crack tip, and this singular term is exactly the same as that constructed in the static case through equations (19–21) of [1]. Thus the problem of determining **h** reduces to a standard elastodynamic problem, whereby the bounded-energy, non-singular part of **h** is identified as the "displacement" field which annihilates the boundary tractions and body forces arising from its known singular part.

Although only transient problems were discussed so far, the results of the present analysis may equally well be applied to study the steady-state response of cracked elastic bodies to harmonically time-varying forcing functions. For example, suppose that the solution corresponding to the loading system $\mathbf{F}^{(i)}(\mathbf{x}, t) = \mathbf{f}^{(i)}(\mathbf{x})e^{i\omega t}$ and $\mathbf{T}^{(i)}(\mathbf{x}, t) = \mathbf{t}^{(i)}(\mathbf{x})e^{i\omega t}$ is given by $\mathbf{U}^{(i)}(\mathbf{x}, t) = \mathbf{u}^{(i)}(\mathbf{x})e^{i\omega t}$. If the solution for loading system 1 is known for all l and ω, then the stress intensity factor resulting from loading system 2, that is, $K^{(2)}(t) = k^{(2)}e^{i\omega t}$, can be determined from (14) with s being replaced by $i\omega$.

Acknowledgement—We wish to acknowledge support of this study by the U.S. Atomic Energy Commission under contract AT(11-1) 3084 with Brown University.

REFERENCES

1. J. R. Rice, Some remarks on elastic crack-tip stress fields, *Int. J. Solids Struct.* **8**, 751–758 (1972).
2. H. F. Bueckner, A novel principle for the computation of stress intensity factors, *Z. angew Math. Mech.* **50**, 529–546 (1970).
3. J. R. Rice, Mathematical analysis in the mechanics of fracture. In *Fracture*, edited by H. Liebowitz, Vol. 2, pp. 191–311. Academic Press (1968).

4. J. K. Knowles and E. Sternberg, On a class of conservation laws in linearized and finite elastostatics, *Arch. ration. Mech. Analysis* **44**, 187–211 (1972).
5. J. D. Eshelby, The energy-momentum tensor in continuum mechanics, in *Inelastic Behavior of Solids*, edited by M. F. Kanninen *et al.* pp. 77–115. McGraw-Hill (1970).
6. F. Nilsson, A path-independent integral for transient crack problems, *Int. J. Solids Struct.* **9**, 1107 (1973).
7. L. B. Freund, The stress intensity factor due to normal impact loading of the faces of a crack, *Int. J. Engng. Sci.* **12**, 179–189 (1974).
8. A. T. deHoop, Representation theorems for the displacement in an elastic solid and their application to elastodynamic diffraction theory, D.Sc. Thesis, Technische Hogeschool, Delft (1958).

Абстракт — Рассматривается линейное упругое тело, в плоском деформированном состоянии, заключающее стационарную трещину, и которое с начала в покое и свободное от напряжений. Доказывается, что когда поля упругодинамического перемещения и факторы интенсивности напряжений известны и представлены в виде функций длины трещины, для любого симметрического распределения усилий, зависящих от времени, которые действуют на это тело и после $t = 0$, так да можно непосредственно определить фактор интенсивности напряжений, вследствие какой-нибудь другой системы симметрической нагрузки, которая действует на тоже самое тело. Другая система нагрузки может быть произвольного пространственного распределения и любого изменения во времени. Далее, можно также непосредственно определить этую часть поля упругодинамического перемещения, вследствие другой системы нагрузки, которая возникает с наличия трещины. Результаты получаются путем обобщения способа выведенного Райсом для соответствующих упругостатических результатов Бюкнера–Райса на преобразование по Лапласу, упругодинамические переменные. Более того, указывается существование всеобщей, упругодинамической «Функции веса», для любого заданного тела с трещинами. В качестве примера, на основе указанного метода, определяется непосредственно последний результат Фрейнда для фактора интенсивности напряжений, вследствие внезапно приложенных, сосредоточенных сил на поверхностях с трещинами. В этом случае, используется более преждевременное решение Гупа, для внезапно приложенных постоянных давлений.